Advanced Mechanics

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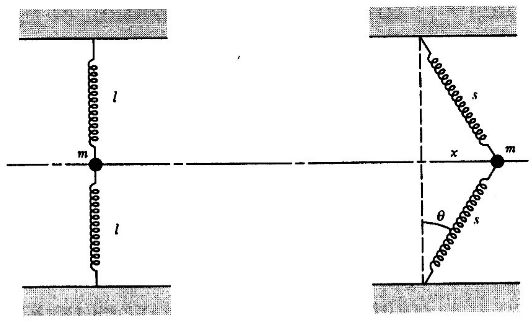
Exam, July 1, 2009.
9:00-12:00, room 5111.0022
5 problems (total of 50 points).
The solution of every problem on a separate piece of paper with name and student number.
Use the attached formula list where necessary.

Problem 1 (10 pnts in total)

Consider the oscillator given in the figure where the unextended length of the springs equals d. The mass m slides over a horizontal bar with a friction force equal to $F_f = -m\beta\dot{x}$ and is subject to an external driving force $F_d = mG\cos\omega t$.



(a) Equilibrium position (b) Extended position

3 pnts a. Show that for sufficiently small $x(t)$ the equation of motion can be written as the real part of $\ddot{x} = -a\,x + \varepsilon x^3 - \beta\dot{x} + Ge^{i\omega t}$ and give the expressions for a and ε.

3 pnts b. Show that for a small driving force the solution can be written approximately as $x(t) = A\,e^{i\omega t} + B\,e^{3i\omega t}$ where $B << A$ and solve for A and B.

2 pnts c. What other frequencies do you expect to appear in the solution for somewhat stronger driving forces (keeping only the 3^{rd} order anharmonic term)?

2 pnts d. Give a schematic sketch of the difference in the phase-space diagram for the the two cases $d = l$ and $d << l$

Problem 2 (10 pnts in total)

Find the shortest path on a surface given by $z = x^{3/2}$ between the two points $P_1 = (0,0,0)$ and $P_2 = (1,1,1)$.

3 pnts a. Express the path length as an integral of a functional over x. Incorporate the constraint in the expression.

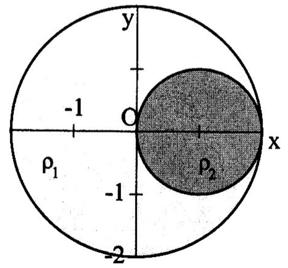
3 pnts b. Show that the solution of the Euler equation can be written as $y(x) = A(1+9x/4)^{3/2}+B$.

1 pnts c. Determine the constants A and B for the path between P_1 and P_2 (do not expect round numbers).

3 pnts d. Determine the path length between P_1 and P_2.

Problem 3 (10 pnts in total)

The object shown in the figure consists of a flat disk with density ρ_1 and radius $R_1 = 2$ to which another disk is attached with density ρ_2 and radius $R_2 = 1$ as indicated. Ignore the thickness of the disks.



3 pnts a. Show that the moment of inertia of a flat disk with mass M and radius R around an axis through its center equals $I = MR^2/2$.

2 pnts b. Calculate the position of the c.m. of the composite system.

5 pnts c. Calculate the moment of inertia around an axis through the c.m., perpendicular to the disks.

Problem 4 (10 pnts in total)

The object given in problem 3 (Moment of inertia around axis through the c.m. is $\frac{R^2}{4}(2M_1 + \frac{M_1 M_2}{M_1+M_2} + M_2/2)$ where M_1 and M_2 are the masses of each of the two disks) is rolling down an inclined slope at an angle α.

4 pnts a. Give the expression for the kinetic energy of the system.

4 pnts b. Give the expression for the potential energy of the system.

2 pnts c. Determine the Euler-Lagrange equation of motion (do not try to find a solution).

Problem 5 (10 pnts in total)

A bullet is shot straight up (along a plumb line) at a latitude λ on Earth. The bullet reaches a height of 500 m and then falls back to Earth. Ignore friction in this problem.

$$\vec{F}' = \vec{F}_{\text{inert}} - 2m\vec{\omega} \times \vec{v}' - m\dot{\vec{\omega}} \times \vec{r}' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}') \ ,$$

$g_0 = 9.8 m/s^2$, $R = 6.371 \times 10^6$ m and assume the Earth to be a perfect sphere.

2 pnts a. Calculate the deviation of a plumb line from a normal to the Earth surface.

2 pnts b. Give the expression for velocity as function of height (ignore corrections due to Earth rotation).

5 pnts c. Calculate the displacement from the plumb line when the bullet has reached the highest point. Give the direction.

1 pnts d. Calculate the horizontal displacement when the bullet has fallen back on Earth (same precision as part c).

$\cos^2\varphi - \sin^2\varphi = \cos^2\varphi + \sin^2\varphi - 2\sin^2\varphi$
$= 1 - 2\sin^2\varphi$

$\sin^2\varphi \sin^2\alpha + \cos^2\varphi \cos^2\alpha =$
$\sin^2\varphi (1 - \cos^2\alpha) + \cos^2\varphi \cos^2\alpha =$
$\sin^2\varphi - \sin^2\varphi \cos^2\alpha + \cos^2\varphi \cos^2\alpha =$
$\sin^2\varphi + \cos^2\alpha (1 - 2\sin^2\varphi)$